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Estimates to correlation decay in discrete dynamical systems by spectral methods

F Brini†§ and S Siboni‡‖

† Dipartimento di fisica, Università degli studi di Bologna, Via Irnerio, 46-40126 Bologna, Italy
‡ Dipartimento di ingegneria dei materiali, Facoltà di Ingegneria, Università degli studi di Trento, Via Mesiano, 77-38050 Trento, Italy

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Abstract. We discuss the use of spectral methods to estimate the decay of correlations of observables in discrete dynamical systems. Some abstract results are stated and applied to the analysis of correlation decay for smooth and analytic observables considered on a particular class of skew-endomorphisms of the 2-torus. General superexponential estimates to correlation decay are also established for the algebraic automorphisms and endomorphisms of the n-torus, whenever hyperbolic or purely expanding conditions occur.

0. Introduction

Decay of correlations for maps plays an important role in statistical physics, particularly in the study of relaxation to equilibrium and in the computation of transport coefficients for dynamical variables like actions, energy or momentum in many systems of physical interest [1]. It also constitutes the first step in order to establish strong statistical properties of measure preserving maps and related observables, such as the central limit theorem and Donsker's invariance principle [2]. Several techniques are available for the numerical and analytical estimate of correlations. Although, in principle, applicable to any kind of maps, numerical methods [3–5] encounter serious limitations due to the possible fast decay of correlations, rapidly hidden by algorithmic and round-off errors, or to the necessarily finite number of obtainable data, which might correspond to a transient behaviour and not be representative of the actual asymptotical trend. On the other hand, the analytical estimates and some semi-analytical approaches [6] are valid only for specific classes of maps, satisfying additional requirements like hyperbolicity [7, 8], 'almost hyperbolicity' in an appropriate sense [9, 10] and so forth. Some constraint on the choice of the observables is also imposed. Most of the previous bounds are obtained by using the concepts of (possibly infinite) Markov partition and symbolic dynamics [11, 12] or suitable generalizations of them [9]. Valid alternative methods have been developed only rather recently [10, 13]. There are some cases, however, where spectral techniques are particularly useful, providing sometimes the only available results or upper bounds related more directly with the system parameters and with the smoothness of the observables.

The present paper is precisely devoted to a digression about the use of spectral methods for the estimate of correlation decay in dynamical systems. We will confine ourselves to

§ E-mail address: brini@bo.infn.it
‖ E-mail address: stefano.siboni@ing.unitn.it

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the case of discrete-time dynamical systems $(T, \Omega, \mathcal{B}, \mu)$, which will be simply defined as a measure preserving transformation T acting on a probability space Ω with probability measure μ and σ-field $\mathcal{B}$. By saying that f is an observable of the dynamical system we mean that f is a real- or complex-valued square-integrable function on Ω with respect to the measure μ. We denote by $L^2(\Omega, \mathcal{B}, \mu)$ the corresponding linear space, where sum and multiplication by a (complex) scalar are defined pointwise and functions which coincide μ-almost everywhere not distinguished. A natural structure of inner product is introduced by means of the integral

$$\langle h|g\rangle = \int_\Omega \overline{h(x)} g(x)\, \mathrm{d}\mu(x) \tag{0.1}$$

for arbitrary functions $h, g \in L^2(\Omega, \mathcal{B}, \mu)$. The mean value of $f \in L^2(\Omega, \mathcal{B}, \mu)$ can then be written as $\langle 1|f\rangle$ by posing $1 \in L^2(\Omega, \mathcal{B}, \mu)$ such that $1(x) = 1$ for μ-almost every $x \in \Omega$. In many typical problems $(T, \Omega, \mathcal{B}, \mu)$ satisfies a strong mixing property, so that for any two observables $f, g \in L^2(\Omega, \mathcal{B}, \mu)$ the correlation function $C_s(f, g) = \langle f|g \circ T^s\rangle - \overline{\langle 1|f\rangle}\langle 1|g\rangle$ tends to zero as $s \to +\infty$. According to the polarization identity this condition occurs if and only if all the correlations of the form $C_s(f, f)$ also decay to zero. Expressions $C_s(f, f)$ are known as the autocorrelations of the observable $f \in L^2(\Omega, \mathcal{B}, \mu)$ and they may also be rewritten as $C_s(f, f) = \langle f|U^s f\rangle - |\langle 1|f\rangle|^2$ in terms of the Koopman operator U associated to T, the linear unitary operator of $L^2(\Omega, \mathcal{B}, \mu)$ defined by $(Uf)(x) = f(T(x))\, \forall x \in \Omega$, $f \in L^2(\Omega, \mathcal{B}, \mu)$. The basic idea which allows us to apply spectral methods to the analysis of correlation decay is founded on a very simple characterization of strong mixing. This states that, given an arbitrary complete orthonormal set $\mathcal{S}$ of $L^2(\Omega, \mathcal{B}, \mu)$, a dynamical system $(T, \Omega, \mathcal{B}, \mu)$ is mixing if and only if correlations between any pair of vectors in $\mathcal{S}$ converge to zero in the limit $s \to \infty$ [14, 15]. Under suitable conditions, information about the rate of correlation decay for vectors of the orthonormal base can be usefully applied to achieve estimates to the correlation decay for various classes of observables. Analogous arguments can also be extended to purely ergodic non-mixing dynamical systems, whenever an infinite orthonormal set of observables is given whose correlations decay suitably fast.

We prove here strong upper bounds to autocorrelations for smooth or analytic observables which can be expanded into a Bessel series of orthonormal $L^2(\Omega, \mathcal{B}, \mu)$ vectors, under the assumption that correlations of orthonormal vectors converge to zero at variously assigned rates. Similar arguments are also applied to the classical algebraic auto- and endomorphisms of the n-torus, generalizing the results in [16] and providing, in particular, superexponential estimates to the autocorrelations of analytic observables, instead of the general exponential bounds already available. Finally, an illustration of the basic ideas on which the above general results about correlation decay rest is briefly given for a class of exact skew-endomorphisms of the 2-torus [17–20]. It is noticeable that for analytical observables the estimated decay is much faster than for smooth ones. This result is interesting not only from a mere mathematical point of view, since although the physical relevance of smooth observables is certainly larger, analytical observables occur in various physical models [21–24], but particularly for applications to particle accelerators [25, 26]. We further remark that the superexponential decay, in the cases where it is proved, would be very difficult to detect by numerical simulations, owing to the high computational precision needed and to the very fast convergence to zero. In fact, we are not aware of physical models where the correlation decay for appropriate observables is satisfactorily fitted by a superexponential law.

The plan of this paper is as follows. In section 1 we state several general results concerning autocorrelation decay of smooth or analytic observables. Section 2 and relative subsections provide the application of spectral methods to algebraic auto- and endomorphisms of the n-torus, showing the superexponential decay of correlations for analytic observables. Section 3 contains a discussion of exact skew-endomorphisms and, finally, section 4 is devoted to some concluding remarks.

1. General results

Let $\mathcal{O}$ be a countable (not necessarily complete) orthonormal set in $L^2(\Omega, \mathcal{B}, \mu)$ and assume that the almost everywhere constant function 1 belongs to $\mathcal{O}$. We consider observables $f \in L^2(\Omega, \mathcal{B}, \mu)$ which can be expanded into a Bessel series of vectors of the orthonormal set $\mathcal{O}$, convergence being intended with respect to the L^2-norm $\|f\|_2 = (\langle f|f\langle)^{1/2}$ induced by the scalar product. Taking $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \mathbb{N}\backslash\{0\}$, we can prove the following theorem 1.1. From a geometrical point of view, the interpretation of the result is rather simple: the map is assumed to induce a transformation of the orthonormal set of characters onto itself, so that Fourier vectors 'mix' in a suitable way. Conditions (1i) and (1ii)—see below—are a formal way to specify how fast this mixing of the character set occurs, in order that the correlations of characters decay at an appropriate rate. This is enough to conclude that, for conveniently regular observables, tight bounds to correlations must hold. The physical meaning of the assumptions depends, of course, on the physical interpretation given to characters, which is in turn system dependent.

Theorem 1.1. Let $\mathcal{O} = \{e_i, i \in \mathbb{N}\}$, with $e_0 = 1$. Suppose there exists a mapping $\tilde{M} : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\tilde{M}(0) = 0$, $\tilde{M}(i) = i$ implies $i = 0$ and $U^s e_i = e_{\tilde{M}^s(i)}$, $\forall s, i \in \mathbb{N}$. Moreover, let $\tilde{M}$ satisfy one of the following properties:

(1i) $\tilde{M}$ is an increasing function of $i \in \mathbb{N}$ obeying the condition $\tilde{M}(i) \geqslant i + 1\, \forall i \in \mathbb{Z}^+$;

(1ii) there exists a constant $\lambda > 1$ such that $\tilde{M}(i) \geqslant \lambda i\, \forall i \in \mathbb{N}$.

Finally, let us consider observables $f \in L^2(\Omega, \mathcal{B}, \mu)$ of the form $f = \sum_{i=0}^{\infty} c_i e_i$ whose Bessel coefficients $c_i \in \mathbb{C}$ tend to zero either exponentially or according to a power law:

(2i) there are constants $\alpha, \beta > 0$ such that $|c_i| \leqslant \alpha\, \mathrm{e}^{-\beta i}\, \forall i \in \mathbb{N}$;

(2ii) there exist $\alpha > 0$ and $\beta > 1/2$ for which $|c_i| \leqslant \alpha i^{-\beta}\, \forall i \in \mathbb{Z}^+$.

Then, by denoting with $C > 0$ a suitable constant:

(a) under conditions (1i) and (2i) we have $|C_s(f, f)| \leqslant C\mathrm{e}^{-\beta s}\, \forall s \in \mathbb{N}$;

(b) conditions (1ii) and (2i) lead to the upper bound $|C_s(f, f)| \leqslant C\mathrm{e}^{-\beta\lambda^s}\, \forall s \in \mathbb{N}$;

(c) conditions (1ii) and (2ii) imply $|C_s(f, f)| \leqslant C\lambda^{-\beta s}\, \forall s \in \mathbb{N}$;

(d) under conditions (1i) and (2ii):

(d1) whenever $\beta > 1$ we have $|C_s(f, f)| \leqslant Cs^{-\beta}\, \forall s \in \mathbb{Z}^+$;

(d2) for $\beta = 1$ there holds $|C_s(f, f) \leqslant Cs^{-1} \log s\, \forall s \in \mathbb{Z}^+$;

(d3) if $\beta \in]1/2, 1[$, for every $\gamma \in]0, 2\beta - 1[$ there exists a real positive sequence $(a_{\gamma,s})_{s\in N}$ such that $|C_s(f, f)| \leqslant a_{\gamma,s}\, s^{-\gamma}\, \forall s \in \mathbb{Z}^+$ and $a_{\gamma,s} \longrightarrow_{s\to\infty} 0$. $\square$

Proof. We preliminary write down a simple general estimate for the autocorrelations. Owing to the continuity of the scalar product we have indeed

$$\langle f|U^s f\rangle = \sum_{i,j=0}^{\infty} \overline{c_j} c_i \int_{\Omega} \overline{e_j(x)} e_{\tilde{M}^s(i)}(x)\, \mathrm{d}\mu(x) = \sum_{i=0}^{\infty} \overline{c_{\tilde{M}^s(i)}} c_i. \tag{1.1}$$

The straightforward upper bound $|C_s(f, f)| \leqslant \sum_{i=1}^{\infty} |c_{\tilde{M}^s(i)}||c_i|$ becomes

$$|C_s(f, f)| \leqslant \alpha^2 \sum_{i=1}^{\infty} \mathrm{e}^{-\beta[i+\tilde{M}^s(i)]} \tag{1.2}$$

in the case of exponential decay of the Bessel coefficients and

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{i=1}^{\infty} \frac{1}{i^\beta} \frac{1}{[\tilde{M}^s(i)]^\beta} \tag{1.3}$$

when a power-decay law for the same coefficients is assumed. Whenever condition (1i) occurs it is easily proved by induction that $\tilde{M}^s(i) \geqslant i+s \; \forall i \in \mathbb{Z}^+$, $s \in \mathbb{N}$, and an analogous result holds when $\tilde{M}$ obeys the bound (1ii): $\tilde{M}^s(i) \geqslant \lambda^s i \; \forall i, s \in \mathbb{N}$.

By using $\tilde{M}^s(i) \geqslant i+s$ within (1.2) we get

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{i=1}^{\infty} \mathrm{e}^{-2\beta i}\, \mathrm{e}^{-\beta s} = \frac{\alpha^2\, \mathrm{e}^{-\beta s}\, \mathrm{e}^{-2\beta}}{1-\mathrm{e}^{-2\beta}}$$

and item (a) is proved. In a similar way, bound $\tilde{M}^s(i) \geqslant \lambda^s i$ puts the estimate (1.2) into the form

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{i=1}^{\infty} \mathrm{e}^{-\beta[i+\lambda^s i]} = \frac{\alpha^2\, \mathrm{e}^{-\beta(1+\lambda^s)}}{1-\mathrm{e}^{-\beta(1+\lambda^s)}}$$

and provides item (b). The same inequality used within (1.3) leads to the upper bound stated as item (c)

$$|C_s(f,f)| \leqslant \sum_{i=1}^{\infty} \alpha i^{-\beta} \alpha i^{-\beta} \lambda^{-\beta s} = \alpha^2 \sum_{i=1}^{\infty} i^{-2\beta} \lambda^{-\beta s}.$$

As for item (d), we have to discuss the estimate $C_s(f,f) \leqslant \alpha^2 \sum_{i=1}^{\infty} i^{-\beta}(i+s)^{-\beta}$. If $\beta > 1$ we can write

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{i=1}^{\infty} \frac{s^\beta}{i^\beta (i+s)^\beta} \frac{1}{s^\beta} \tag{1.4}$$

and consider the sequence of functions on $i \in \mathbb{Z}^+$ given by $\Phi_s(i) = s^\beta i^{-\beta}(i+s)^{-\beta}$, with $s \in \mathbb{N}$. A straightforward calculation shows that $\lim_{s\to+\infty} \Phi_s(i) = i^{-\beta} \; \forall i \in \mathbb{Z}^+$, whereas, uniformly on s, $|\Phi_s(i)| \leqslant i^{-\beta} \; \forall i \in \mathbb{Z}^+$. Finally, $\sum_{i=1}^{\infty} i^{-\beta} < +\infty$. By Lebesgue's dominated convergence theorem we deduce the existence of the limit

$$\lim_{s\to+\infty} \sum_{i=1}^{\infty} \frac{s^\beta}{i^\beta (i+s)^\beta} = \sum_{i=1}^{\infty} \frac{1}{i^\beta} < +\infty \tag{1.5}$$

and, by denoting with $C > 0$ a suitable constant, the consequent bound $\sum_{i=1}^{\infty} s^\beta i^{-\beta} (i+s)^{-\beta} \leqslant C\alpha^{-2} \; \forall s \in \mathbb{N}$. This result, together with (1.4), provides (d1).

For $\beta = 1$ we obtain $|C_s(f,f)| \leqslant \alpha^2 \sum_{i=1}^{\infty} [i(i+s)]^{-1} = \alpha^2 s^{-1} \sum_{i=1}^{s} i^{-1}$ and since there exists $C > 0$ satisfying

$$\frac{1}{s} \sum_{i=1}^{s} \frac{1}{i} \leqslant \frac{1}{s}\left[2 + \frac{1}{\log 2}\log s\right] \leqslant \frac{C}{\alpha^2}\frac{1}{s}\log s \qquad \forall s \in \mathbb{Z}^+ \tag{1.6}$$

formula (d2) is established. When $\beta \in]1/2, 1[$ let us consider ε such that $0 < \varepsilon < \beta$. It is easy to verify that $\forall i \in \mathbb{Z}^+$ the function $\Psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Psi_i(s) = s^{\beta-\varepsilon}(i+s)^{-\beta}$ takes its maximum at $s = i(\beta-\varepsilon)/\varepsilon$ and consequently

$$\operatorname*{Sup}_{s\in\mathbb{Z}^+} \left| \frac{s^{\beta-\varepsilon}}{(i+s)^\beta} \right| \leqslant \left(\frac{\beta}{\varepsilon} - 1\right)^{\beta-\varepsilon} \left(\frac{\varepsilon}{\beta}\right)^{\beta} \frac{1}{i^\varepsilon}. \tag{1.7}$$

Therefore, for the series $\sum_{i=1}^{\infty} i^{-\beta}\Psi_i(s)$ we have that

- for every fixed $\varepsilon \in]0, \beta[$ and $\forall i \in \mathbb{Z}^+$ there holds $\lim_{s\to+\infty} i^{-\beta}\Psi_i(s) = 0$;

- the upper bound below is satisfied

$$\left|\frac{1}{i^\beta}\Psi_i(s)\right| \leqslant \frac{1}{i^\beta}\operatorname*{Sup}_{s\in\mathbb{Z}^+}\left|\frac{s^{\beta-\varepsilon}}{(i+s)^\beta}\right| \leqslant \left(\frac{\beta}{\varepsilon}-1\right)^{\beta-\varepsilon}\left(\frac{\varepsilon}{\beta}\right)^\beta \frac{1}{i^{\beta+\varepsilon}} \tag{1.8}$$

uniform with respect to s and integrable with respect to $i \in \mathbb{Z}^+$, provided that $\beta + \varepsilon > 1$.

By the dominated convergence theorem we deduce that $\lim_{s\to+\infty}\sum_{i=1}^{\infty} i^{-\beta}\Psi_i(s) = 0$. The above conclusion is valid under the assumption that the value of ε matches the inequalities $0 < \varepsilon < \beta$ and $1 < \beta + \varepsilon$, with $1/2 < \beta < 1$. For a given β a simple calculation shows this happens if and only if $\beta - \varepsilon \in]0, 2\beta - 1[$. Statement (d3) follows by posing $\gamma = \beta - \varepsilon$ and $a_{\gamma,s} = \sum_{i=1}^{\infty} s^\gamma i^{-\beta}(i+s)^{-\beta}$. The proof is complete. □

A more general result is given by the following statement.

Theorem 1.2. Let $\mathcal{O}$ be an orthonormal set of the form $\{e_{0,0} = 1\} \cup \{e_{k,i} : i \in \mathbb{Z}^+, k \in D \subseteq \mathbb{N}\}$ and suppose that there exists a mapping $\tilde{M} : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $Ue_{k,i} = e_{k,\tilde{M}(i)}\ \forall k \in D$ and $\forall i \in \mathbb{N}$. Moreover, let $\tilde{M}$ satisfy one of the same hypotheses (1i) and (1ii) stated in theorem 1.1 and let $f \in L^2(\Omega, \mathcal{B}, \mu)$ be any observable expressed by a series

$$f = \sum_{k\in D}\sum_{i=1}^{\infty} c_{k,i}e_{k,i} + c_{0,0} \qquad c_{k,i}, c_{0,0} \in \mathbb{C} \tag{1.9}$$

whose Bessel coefficients tend to zero either exponentially or according to a power law:

(2i′) there exist $\alpha, \beta > 0$ such that $|c_{k,i}| \leqslant \alpha\, \mathrm{e}^{-\beta(k+i)}\ \forall k \in D, i \in \mathbb{Z}^+$;

(2ii′) for suitable constants $\alpha > 0$ and $\beta > 1$ there holds $|c_{k,i}| \leqslant \alpha(k+i)^{-\beta}\ \forall k \in D$, $i \in \mathbb{Z}^+$.

Then, with a suitable constant $C > 0$:

(a) if conditions (1i) and (2i′) occur, the bound $|C_s(f, f)| \leqslant C\, \mathrm{e}^{-\beta s}$ holds $\forall s \in \mathbb{N}$;

(b) when both (1ii) and (2i′) hold we have $|C_s(f, f)| \leqslant C\, \mathrm{e}^{-\beta\lambda^s}\ \forall s \in \mathbb{N}$;

(c) under conditions (1ii) and (2ii′) we deduce $|C_s(f, f)| \leqslant C\lambda^{-\beta s}\ \forall s \in \mathbb{N}$;

(d) conditions (1i) and (2ii′) imply that

(d1) whenever $\beta > 2$ we get $|C_s(f, f)| \leqslant Cs^{-\beta}\ \forall s \in \mathbb{Z}^+$;

(d2) for $\beta = 2$, define the norm $\|x\|_m$ of a vector $x \in \mathbb{R}^2$ as $\|x\|_m = \max_i |x_i|$, where x_1, x_2 are the components of x with respect to the canonical basis in $\mathbb{R}^2$. Moreover, for each $r \in \mathbb{Z}^+$ let $N_D(r)$ be the cardinality of the set

$$\{(k, i) : k \in D, i \in \mathbb{Z}^+\} \cap \{w \in \mathbb{Z}^2 : \|w\|_m = r\}. \tag{1.10}$$

For every $s \in \mathbb{Z}^+$ we have then the inequalities $|C_s(f, f)| \leqslant C\, s^{-2}$, if there exists $\eta \in]0, 1[$ such that $N_D(r) \sim r^\eta$ as $r \to +\infty$, and $|C_s(f, f)| \leqslant C\, s^{-2}\log s$, whenever $N_D(r) \sim r$ as $r \to +\infty$;

(d3) if $\beta \in]1, 2[$, for every $\gamma \in]0, 2(\beta - 1)[$ there exists a real positive sequence $(a_{\gamma,s})_{s\in N}$ such that $|C_s(f, f)| \leqslant a_{\gamma,s}(1/s^\gamma)\ \forall s \in \mathbb{Z}^+$ and $a_{\gamma,s} \longrightarrow_{s\to+\infty} 0$. □

Proof. We firstly take $C_s(f, f) = \langle f|U^s f\rangle - \langle f|1\rangle\langle 1|f\rangle = \sum_{k\in D}\sum_{i=1}^{\infty} \overline{c_{k,\tilde{M}^s(i)}}c_{k,i}$ and write the general upper bound $|C_s(f, f)| \leqslant \sum_{k\in D}\sum_{i=1}^{\infty} |c_{k,\tilde{M}^s(i)}||c_{k,i}|\ \forall s \in \mathbb{N}$. The last inequality reduces to

$$|C_s(f, f)| \leqslant \alpha^2 \sum_{k\in D}\sum_{i=1}^{\infty} \mathrm{e}^{-2\beta k}\, \mathrm{e}^{-\beta(i+\tilde{M}^s(i))} \tag{1.11}$$

in the case of an exponential decay of Bessel coefficients and

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{k\in D}\sum_{i=1}^{\infty} \frac{1}{(k+i)^\beta}\frac{1}{(k+\tilde{M}^s(i))^\beta} \tag{1.12}$$

when a power-decay law for the same coefficients is assumed.

By inserting $\tilde{M}^s(i) \geqslant i+s$ within (1.11) we readily achieve item (a)

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{k\in D}\sum_{i=1}^{\infty} \mathrm{e}^{-2\beta k}\,\mathrm{e}^{-\beta(2i+s)} \leqslant \mathrm{e}^{-\beta s}\left[\frac{\alpha\,\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-2\beta}}\right]^2 \tag{1.13}$$

whereas the use of $\tilde{M}^s(i) \geqslant \lambda^s i$ inside (1.11) leads to item (b)

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{k\in D}\sum_{i=1}^{\infty} \mathrm{e}^{-2\beta k}\,\mathrm{e}^{-\beta(i+\lambda^s i)} \leqslant \frac{\alpha^2\,\mathrm{e}^{-\beta}}{(1-\mathrm{e}^{-2\beta})^2}\,\mathrm{e}^{-\beta\lambda^s}. \tag{1.14}$$

In the case (c) an upper bound to autocorrelations is

$$|C_s(f,f)| \leqslant \sum_{k\in D}\sum_{i=1}^{\infty} \frac{\alpha^2}{(k+i)^\beta(k+\lambda^s i)^\beta} \leqslant \sum_{k\in D}\sum_{i=1}^{\infty} \frac{\alpha^2}{(k+1)^\beta i^\beta}\frac{1}{\lambda^{\beta s}} \leqslant \left[\sum_{i=1}^{\infty}\frac{\alpha}{i^\beta}\right]^2\frac{1}{\lambda^{\beta s}} \tag{1.15}$$

the residual series being convergent because of the hypothesis $\beta > 1$.

As for item (d), inequality (1.12), together with $\tilde{M}^s(i) \geqslant i+s$, gives

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{k\in D}\sum_{i=1}^{\infty} \frac{1}{(k+i)^\beta}\frac{1}{(k+i+s)^\beta}. \tag{1.16}$$

If $\beta > 2$ let us introduce the function $\Gamma_s(k,i) = s^\beta(k+i)^{-\beta}(k+i+s)^{-\beta}$ and notice that, uniformly on $s \in \mathbb{N}$, $|\Gamma_s(k,i)| \leqslant \lim_{s\to+\infty}\Gamma_s(k,i) = (k+i)^{-\beta}$ with

$$\sum_{k\in D}\sum_{i=1}^{\infty}\frac{1}{(k+i)^\beta} \leqslant \sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(k+i)^\beta} \leqslant \sum_{r=1}^{\infty}\frac{2r}{(r+1)^\beta} < +\infty. \tag{1.17}$$

By dominated convergence we deduce the existence of

$$\lim_{s\to+\infty}\sum_{k\in D}\sum_{i=1}^{\infty}\Gamma_s(k,i) = \sum_{k\in D}\sum_{i=1}^{\infty}\frac{1}{(k+i)^\beta} < +\infty \tag{1.18}$$

so that the sequence $\sum_{k\in D}\sum_{i=1}^{\infty}\Gamma_s(k,i)$ is bounded by a suitable constant $C/\alpha^2 > 0$ independent on s and inequality (d1) holds.

Case $\beta = 2$ is better analysed in a direct way, by means of the bound

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{k\in D}\sum_{i=1}^{\infty}\frac{1}{(k+i)^2(k+i+s)^2} = \alpha^2\sum_{k\in D}\sum_{i=1}^{\infty}\left[\frac{1}{k+i}-\frac{1}{k+i+s}\right]^2\frac{1}{s^2}. \tag{1.19}$$

Notice that

$$\mathcal{F}(s) = \sum_{k\in D}\sum_{i=1}^{\infty}\left[\frac{1}{k+i}-\frac{1}{k+i+s}\right]^2$$

is an increasing function of $s \in \mathbb{N}$. In particular, $\mathcal{F}(0) = 0$ and by Levi's theorem there exists

$$\lim_{s\to+\infty}\mathcal{F}(s) = \sum_{k\in D}\sum_{i=1}^{\infty}\frac{1}{(k+i)^2}. \tag{1.20}$$

The series on the right-hand side of the above equality is not necessarily convergent, but recalling the definition of $N_D(r)$ we can write

$$\frac{1}{2^\beta}\sum_{r=1}^{\infty}\frac{N_D(r)}{r^\beta} = \sum_{r=1}^{\infty}\frac{1}{(2r)^\beta}N_D(r) \leqslant \sum_{k\in D}\sum_{i=1}^{\infty}\frac{1}{(k+i)^\beta} \leqslant \sum_{r=1}^{\infty}\frac{1}{r^\beta}N_D(r) \tag{1.21}$$

and conclude that the convergence of $\sum_{k\in D}\sum_{i=1}^{\infty}(k+i)^{-2}$ depends on the asymptotical behaviour of $N_D(r)$. If, for instance, D were finite—a somehow trivial case, reducible to previous theorem 1.1—for every $r \in \mathbb{N}$ large enough we should have $N_D(r) = \#D < +\infty$ and the convergence of series (1.20) would occur if and only if $\#D\sum_{r=1}^{\infty} r^{-\beta} < +\infty$. On having actually $\beta = 2$, we would conclude that $\mathcal{F}(s) \leqslant \lim_{s\to+\infty}\mathcal{F}(s) = \mathcal{F}(+\infty) < +\infty$, where $\mathcal{F}(+\infty)$ stands for the finite limit (1.20). The more interesting situation of a countable D and $N_D(r) \sim r^\eta$ as $r \to +\infty$, for some $\eta \in]0, 1[$, can be solved in the same way since inequality $\mathcal{F}(+\infty) < +\infty$ still holds and consequently $|C_s(f, f)| \leqslant \alpha^2\mathcal{F}(+\infty)s^{-2}\,\forall s \in \mathbb{Z}^+$. If $N_D(r) \sim 2r(r \to +\infty)$—and the asymptotic behaviour of $N_D(r)$ could not be more rapidly increasing than this, because of the trivial inequality $N_D(r) \leqslant 2r\,\forall r \in \mathbb{Z}^+$—the previous method fails due to $\mathcal{F}(+\infty) = +\infty$. To overcome this difficulty we make use of the identity

$$(k+i)^{-2}(k+i+s)^{-2} = [(k+i)^{-2} - (k+i+s)^{-2}][s^2 + 2(k+i)s]^{-1}$$

which replaced inside (1.19) gives

$$|C_s(f, f)| \leqslant \frac{\alpha^2}{s^2}\lim_{N\to+\infty}\sum_{k=0}^{N}\left[\sum_{i=1}^{N}\frac{1}{(k+i)^2} - \sum_{i=1+s}^{N+s}\frac{1}{(k+i)^2}\right]. \tag{1.22}$$

By taking $N > s$ we can rewrite the last term in (1.22) as

$$\frac{1}{s^2}\alpha^2\lim_{N\to+\infty}\sum_{k=0}^{N}\left[\sum_{i=1}^{s}\frac{1}{(k+i)^2} - \sum_{i=N+1}^{N+s}\frac{1}{(k+i)^2}\right] = \frac{1}{s^2}\alpha^2\lim_{N\to+\infty}\sum_{k=0}^{N}\sum_{i=k+1}^{k+s}\frac{1}{i^2} \tag{1.23}$$

and since

$$\lim_{N\to+\infty}\sum_{k=0}^{N}\sum_{i=k+1}^{k+s}\frac{1}{i^2} \leqslant \sum_{k=1}^{\infty}\frac{1}{k^2} + \lim_{N\to+\infty}\left[\sum_{k=1}^{s-1}\frac{1}{k} - \sum_{k=2+N}^{s+N}\frac{1}{k}\right] = \sum_{k=1}^{\infty}\frac{1}{k^2} + \sum_{k=1}^{s-1}\frac{1}{k} \tag{1.24}$$

there will exist a constant $A > 0$ such that

$$|C_s(f, f)| \leqslant \frac{1}{s^2}\alpha^2\left[\frac{\pi^2}{6} + \sum_{k=1}^{s-1}\frac{1}{k}\right] \leqslant A\frac{1}{s^2}\log s \qquad \forall s \in \mathbb{Z}^+ \tag{1.25}$$

which completes the proof of item (d2).

Let us finally discuss the case $\beta \in]1, 2[$. We preliminarily choose $\varepsilon \in]0, \beta[$ and write

$$|C_s(f, f)| \leqslant \frac{\alpha^2}{s^{\beta-\varepsilon}}\sum_{k\in D}\sum_{i=1}^{\infty}\frac{1}{(k+i)^\beta}\frac{s^{\beta-\varepsilon}}{(k+i+s)^\beta}. \tag{1.26}$$

For every fixed pair (k, i) let us define the function $\Delta_{k,i} : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\Delta_{k,i}(s) = s^{\beta-\varepsilon}(k+i+s)^{-\beta}$ and compute its supremum. A trivial calculation provides the maximum $s^* = (k+i)(\beta-\varepsilon)/\varepsilon \in \mathbb{R}^+$ and whence

$$\operatorname*{Sup}_{s\in\mathbb{Z}^+}|\Delta_{k,i}(s)| \leqslant \Delta_{k,i}(s^*) = \left(\frac{\beta}{\varepsilon} - 1\right)^{\beta-\varepsilon}\left(\frac{\varepsilon}{\beta}\right)^\beta\frac{1}{(k+i)^\varepsilon}. \tag{1.27}$$

Consider the series in (1.26). It is straightforward to show that $\forall(k,i)$ there exists $\lim_{s\to+\infty}(k+i)^{-\beta}\Delta_{k,i}(s)=0$ and that the upper bound, uniform on $s\in\mathbb{N}$,

$$\left|\frac{1}{(k+i)^{\beta}}\Delta_{k,i}(s)\right| \leqslant \left(\frac{\beta}{\varepsilon}-1\right)^{\beta-\varepsilon}\left(\frac{\varepsilon}{\beta}\right)^{\beta}\frac{1}{(k+i)^{\beta+\varepsilon}} \tag{1.28}$$

is integrable over $(k,i)\in D\times\mathbb{Z}^{+}$ whenever $\beta+\varepsilon>2$. By dominated convergence we deduce $\lim_{s\to+\infty}\sum_{k\in D}\sum_{i=1}^{\infty}(k+i)^{-\beta}\Delta_{k,i}(s)=0$. For any fixed $\beta\in]1,2[$ the useful values of ε are those of the interval $]2-\beta,\beta[$ and therefore $\beta-\varepsilon\in]0,2(\beta-1)[$. We simply have to pose $\gamma=\beta-\varepsilon$ and $a_{\gamma,s}=\sum_{k\in D}\sum_{i=1}^{\infty}(k+i)^{-\beta}\Delta_{k,i}(s)$ to obtain (d3) and complete the proof of theorem 1.2. □

Remark. It is clear that under the hypotheses of theorems 1.1 and 1.2, the case of a finite orthonormal set trivially implies autocorrelations of any observable to be definitively zero as $s\to+\infty$.

The conditions assumed by theorems 1.1 and 1.2 essentially mean that the Koopman operator U yields a suitable rearrangement of the orthonormal set $\mathcal{O}$, so that correlations between orthonormal vectors in $\mathcal{O}$ can take two possible values only, either 0 or 1. These conditions could be matched, for instance, when the dynamical system T has a Lebesgue spectrum [14, 27]. A more interesting situation to be discussed is that where the action of U does not merely reduce to a simple vector exchange and consequently all of the correlations of vectors in $\mathcal{O}$ give a non-trivial contribution to the autocorrelations of the observable f. We have, in particular, the following statement.

Theorem 1.3. Let $\mathcal{O}=\{e_i, i\in\mathbb{N}\}$, with $e_0=1$. Suppose $\forall i,j\in\mathbb{N}$ there exist $\alpha_{ij}>0$ and $\rho_{ij}\in]0,1[$ such that

$$|\langle e_i|U^s e_j\rangle-\langle e_i|e_0\rangle\langle e_0|e_j\rangle|\leqslant\alpha_{ij}\rho_{ij}^{s}\qquad\forall s\in\mathbb{N} \tag{1.29}$$

and let f be any $L^2(\Omega,\mathcal{B},\mu)$ function expandable into a Bessel series $f=\sum_{i=0}^{\infty}c_ie_i$ of $\mathcal{O}$ whose coefficients decay either exponentially, $|c_i|\leqslant A\,\mathrm{e}^{-\beta i}\,\forall i\in\mathbb{N}$, or according to a power law, $|c_i|\leqslant Ai^{-\beta}\,\forall i\in\mathbb{Z}^{+}$ (for some constants $A,\beta>0$). Let us denote with $F(i)$ the functions $\mathrm{e}^{-\beta i}$ and $i^{-\beta}$, in the first and in the second case, respectively. Then:

(a) if there exists $\rho\in]0,1[$ such that $\rho_{ij}\leqslant\rho\,\forall(i,j)\in\mathbb{Z}^{+}\times\mathbb{Z}^{+}$ and $\sum_{ij=1}^{\infty}\alpha_{ij}F(i)F(j)<+\infty$, the correlation decay of f is exponential, with rate ρ;

(b) if a $\gamma>0$ can be chosen for which the series

$$\sum_{ij=1}^{\infty}\alpha_{ij}\exp\{-\gamma\log(-\log\rho_{ij})\}F(i)F(j) \tag{1.30}$$

turns out to be convergent, there exists a non-negative sequence $(a_{\gamma,s})_{s\in\mathbb{Z}^{+}}$ satisfying $|C_s(f,f)|\leqslant a_{\gamma,s}s^{-\gamma}\,\forall s\in\mathbb{Z}^{+}$ and $a_{\gamma,s}\longrightarrow_{s\to+\infty}0$. □

Proof. Consider $F(i)=\mathrm{e}^{-\beta i}$. The statement follows from the inequality

$$|C_s(f,f)|\leqslant\sum_{ij=1}^{\infty}|c_i||c_j|\alpha_{ij}\rho_{ij}^{s}\leqslant A^2\sum_{ij=1}^{\infty}\mathrm{e}^{-\beta(i+j)}\alpha_{ij}\rho_{ij}^{s}\qquad\forall s\in\mathbb{N} \tag{1.31}$$

which immediately provides item (a): $|C_s(f,f)|\leqslant A^2\sum_{ij=1}^{\infty}\mathrm{e}^{-\beta(i+j)}\alpha_{ij}\rho^{s}\,\forall s\in\mathbb{N}$. As for item (b), let us rewrite (1.31) in the equivalent form

$$|C_s(f,f)|\leqslant\frac{A^2}{s^{\gamma}}\sum_{ij=1}^{\infty}\mathrm{e}^{-\beta(i+j)}s^{\gamma}\rho_{ij}^{s} \tag{1.32}$$

notice that $\forall i, j \in \mathbb{Z}^+$ there exists $\lim_{s\to+\infty} \mathrm{e}^{-\beta(i+j)}\alpha_{ij}s^\gamma \rho_{ij}^s = 0$ and consider the upper bound

$$\mathrm{e}^{-\beta(i+j)}\alpha_{ij}s^\gamma \rho_{ij}^s \leqslant \mathrm{e}^{-\beta(i+j)}\alpha_{ij} \operatorname*{Sup}_{s\in N}[s^\gamma \rho_{ij}^s]. \tag{1.33}$$

We have that the function $\mathbb{Z}_{ij} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by $\mathbb{Z}_{ij}(s) = s^\gamma \rho_{ij}^s$ tends to zero as $s \to +\infty$ and its maximum on $\mathbb{R}^+$ is readily calculated as $s^* = -\gamma/\log \rho_{ij}$. Thus

$$\mathbb{Z}_{ij}(s^*) = \left(-\frac{\gamma}{\log \rho_{ij}}\right)^\gamma \rho_{ij}^{-\gamma/\log \rho_{ij}} = \gamma^\gamma \mathrm{e}^{-\gamma}(-\log \rho_{ij})^{-\gamma}. \tag{1.34}$$

Inequality (1.33) is then replaced by the following one, uniform on s,

$$\mathrm{e}^{-\beta(i+j)}\alpha_{ij}s^\gamma \rho_{ij}^s \leqslant \mathrm{e}^{-\beta(i+j)}\alpha_{ij} \operatorname*{Sup}_{s\in R^+}[s^\gamma \rho_{ij}^s] \leqslant \gamma^\gamma \mathrm{e}^{-\gamma}\alpha_{ij} \exp\{-\beta(i+j) - \gamma \log(-\log \rho_{ij})\} \tag{1.35}$$

for which hypothesis (1.30) assures

$$\sum_{ij=1}^{\infty} \mathrm{e}^{-\beta(i+j)}\alpha_{ij}s^\gamma \rho_{ij}^s \leqslant \gamma^\gamma \mathrm{e}^{-\gamma} \sum_{ij=1}^{\infty} \alpha_{ij} \exp\{-\beta(i+j) - \gamma \log(-\log \rho_{ij})\} < +\infty. \tag{1.36}$$

Dominated convergence now provides $\lim_{s\to+\infty} \sum_{ij=1}^{\infty} \mathrm{e}^{-\beta(i+j)}\alpha_{ij}s^\gamma \rho_{ij}^s = 0$ and item (b) follows by simply taking $a_{\gamma,s} = A^2 \sum_{ij=1}^{\infty} \mathrm{e}^{-\beta(i+j)}\alpha_{ij}s^\gamma \rho_{ij}^s$.

The proof of case $F(i) = i^{-\beta}$ is completely analogous. □

We can specialize the general statements of theorem 1.3 to a particular situation which will occur in the analysis of correlation decay for a certain class of mixing skew-systems—see section 3. We have the following.

Proposition 1.4. Let $\varphi : \mathbb{R} \longrightarrow]0, 1]$ be a real function such that:

(i) φ is continuous and periodic with period 1 on $\mathbb{R}$;

(ii) $\varphi(x) = 1$ if and only if $x \in \mathbb{Z}$;

(iii) the restriction of φ to a neighbourhood of $x = 0$ is C^1;

(iv) φ admits a negative second derivative in $x = 0$, $\varphi''(0) < 0$.

Consider a given vector $(a, b) \in \mathbb{R}^2\backslash\{(0, 0)\}$ satisfying the 'Diophantine-like' condition†

$$[\mathrm{dist}(ai + bj, \mathbb{Z})]^{-1} \leqslant \mu_{a,b}(i + j)^{\gamma_{a,b}} \qquad \forall (i, j) \in \mathbb{N} \times \mathbb{N}\backslash\{(0, 0)\} \tag{1.37}$$

for some constants $\mu_{a,b}, \gamma_{a,b} > 0$, when having defined $\mathrm{dist}(ai+bj, \mathbb{Z}) = \mathrm{Inf}_{q\in Z}|ai+bj-q|$. Suppose that $\mathcal{O}$ is the same orthonormal set of theorem 1.3 and that inequality (1.29) holds, with the factors α_{ij} uniformly bounded: $\alpha_{ij} \leqslant K\ \forall i, j \in \mathbb{N}$ and

$$\rho_{ij} = \varphi(ai + bj) \qquad \forall (i, j) \in \mathbb{N} \times \mathbb{N}\backslash\{(0, 0)\}. \tag{1.38}$$

Finally, let f be an observable of the form $f = \sum_{i=0}^{\infty} c_i e_i$, $e_i \in \mathcal{O}$.

Then, with the same notation introduced in theorem 1.3:

(a) if the Bessel coefficients c_i of f decrease exponentially to zero, for every $\gamma > 0$ there exists a non-negative sequence $(a_{\gamma,s})_{s\in\mathbb{Z}^+}$ such that $|C_s(f, f)| \leqslant a_{\gamma,s}s^{-\gamma}\ \forall s \in \mathbb{Z}^+$ and $a_{\gamma,s} \longrightarrow_{s\to+\infty} 0$;

(b) the same property is verified for every γ of the type $0 < \gamma < (\beta - 2)/(2\gamma_{a,b})$ whenever the Bessel coefficients of f obey a power-decay law and provided that $\beta > 2$.

† Which is certainly satisfied if, for instance, $a = 0$ and b is Diophantine in the usual sense, or *vice versa*.

Proof. Owing to periodicity we can confine ourselves to consider the restriction of φ to the interval $[-1/2, 1/2]$. Let $\delta \in]0, 1/2[$ such that the restriction of φ to $]-\delta, \delta[$ is C^1. According to Peano's version of the Taylor formula, $\forall x \in]-\delta, \delta[$ we can write

$$\varphi(x)^{-1} - 1 = \left(-\frac{\varphi''(0)}{2} + o(1)\right) x^2 \qquad (x \to 0). \tag{1.39}$$

Since $o(1) \to 0$ as $x \to 0$, for every $h_\delta \in]0, -\varphi''(0)/2[$ it is always possible to choose a (possibly smaller) δ such that

$$\varphi(x)^{-1} - 1 \geqslant \left(-\frac{\varphi''(0)}{2} - h_\delta\right) x^2 \qquad \forall x \in]-\delta, \delta[\tag{1.40}$$

and simultaneously

$$\Delta_\delta = \left(-\frac{\varphi''(0)}{2} - h_\delta\right) \delta^2 < 1/2.$$

For every $x \in [-1/2, 1/2]\backslash]-\delta, \delta[$ we have instead $\varphi(x)^{-1} - 1 \geqslant H_\delta^{-1} - 1$, where $H_\delta = \max\{\varphi(\xi) : \xi \in [-1/2, 1/2]\backslash]-\delta, \delta[\}$ certainly belongs to the interval $]0, 1[$ owing to the properties of φ. From (1.40) we derive the lower bound

$$-\log\varphi(x) \geqslant \log\left[1 + \left(-\frac{\varphi''(0)}{2} - h_\delta\right) x^2\right]. \tag{1.41}$$

But $\forall z \in [0, \Delta_\delta]$, $\Delta_\delta < 1/2$, there holds $\log(1+z) \geqslant Q_\delta z$ with

$$Q_\delta = 1 - \frac{\Delta_\delta}{2(1-\Delta_\delta)^2} > 0$$

and since $(-\varphi''(0)/2 - h_\delta)x^2 \leqslant \Delta_\delta \, \forall x \in]-\delta, \delta[$, inequality (1.41) can be replaced by

$$-\log\varphi(x) \geqslant Q_\delta \left(-\frac{\varphi''(0)}{2} - h_\delta\right) x^2$$

which implies

$$\exp\left[-\gamma \log(-\log\varphi(x))\right] \leqslant \left[Q_\delta\left(-\frac{\varphi''(0)}{2} - h_\delta\right)\right]^{-\gamma} \frac{1}{x^{2\gamma}}. \tag{1.42}$$

The appropriate bound for $x \in [-1/2, 1/2]\backslash]-\delta, \delta[$ is much simpler

$$\exp[-\gamma \log(-\log\varphi(x))] \leqslant (-\log H_\delta)^{-\gamma}. \tag{1.43}$$

Recalling (1.38), inequality (1.42) can be rewritten as

$$\exp\{-\gamma \log(-\log\rho_{ij})\} \leqslant \left[Q_\delta\left(-\frac{\varphi''(0)}{2} - h_\delta\right)\right]^{-\gamma} [\mathrm{dist}(ai + bj, \mathbb{Z})]^{-2\gamma} \tag{1.44}$$

and with the obvious definition of the constant $R > 0$, condition (1.37) leads to the basic upper bound

$$\exp\{-\gamma \log(-\log\rho_{ij})\} \leqslant R(i+j)^{2\gamma\gamma_{a,b}} \tag{1.45}$$

valid whenever $\mathrm{dist}(ai + bj, \mathbb{Z}) < \delta$, whereas (1.43) applies to any other case. Condition (1.30) of theorem 1.3 is certainly satisfied for any choice of $\gamma > 0$ because of the factor $\mathrm{e}^{-\beta(i+j)}$, which proves item (a).

In an analogous way, in order to check the convergence of (1.30) for $F(i) = i^{-\beta}$ we write

$$\sum_{ij=1}^{\infty} \frac{\alpha_{ij}}{i^\beta j^\beta} \exp\{-\gamma \log(-\log\rho_{ij})\} \leqslant K(-\log H_\delta)^{-\gamma}\left[\sum_{i=1}^{\infty} \frac{1}{i^\beta}\right]^2 + \sum_{ij=1}^{\infty} \frac{KR}{i^\beta j^\beta}(i+j)^{2\gamma_{a,b}\gamma}. \tag{1.46}$$

The first term on the right-hand side of (1.46) converges since, according to our hypotheses, $\beta > 2$. As for the latter series, on having $xy \geqslant x + y\ \forall x, y \geqslant 2$ we can achieve the estimate

$$\sum_{ij=1}^{\infty} \frac{1}{i^\beta j^\beta}(i+j)^{2\gamma_{a,b}\gamma} = 4^\beta \sum_{ij=1}^{\infty} \frac{1}{(2i \cdot 2j)^\beta}(i+j)^{2\gamma_{a,b}\gamma} \leqslant 2^\beta \sum_{ij=1}^{\infty} \frac{1}{(i+j)^{\beta-2\gamma_{a,b}\gamma}} \tag{1.47}$$

convergent provided that $\gamma < (\beta-2)/(2\gamma_{a,b})$, where $(\beta-2)/(2\gamma_{a,b}) > 0$. As a result, for every $\gamma \in]0, (\beta-2)/(2\gamma_{a,b})[$ theorem 1.3 can be applied. The proof of item (b) is complete. $\square$

2. Algebraic automorphisms and endomorphisms of the n-torus $\mathbb{T}^n$

In this section we apply spectral methods to estimating correlation decay in algebraic auto- and endo-morphisms of the n-torus. The results constitute a generalization of estimates given in [20, 28] for the cat map [27]. They must not be simply considered as an immediate application of the statements in section 1 (theorems 1.1 and 1.2, in particular). Nevertheless, the general idea which allows one to prove the superexponential correlation decay of analytical observables for the algebraic auto/endomorphisms of the torus is essentially the same as that on which theorems 1.1 and 1.2 are based: the map induces a transformation of the orthonormal set of characters onto itself, so that Fourier vectors 'mix' appropriately. Here the situation is only a little more complicated and requires, as will be clear in the following, a 'pairwise' treatment of the Fourier vectors in order to obtain the improved estimates.

Preliminarily, we recall some basic definitions and introduce the notation which will be used from now on. By $\mathbb{T}^n$ we denote the n-dimensional torus, parametrized by the unit cube $[0, 1[^n$. As usual, $\mathbb{T}^n$ is intended to be equipped with the Lebesgue–Haar measure μ on the σ-field $\mathcal{B}$ of Borel sets in $[0, 1[^n$ and takes the structure of a probability space. The n-torus can also be thought of as the quotient of the whole $\mathbb{R}^n$ with respect to the equivalence relation which defines as equivalent two elements of $\mathbb{R}^n$ whose coordinates in the canonical base differ by integers. In this context $\mathbb{R}^n$ is also referred to as the covering space of the torus. The covering map $x' = x \bmod [0, 1[^n$ associates to any $x \in \mathbb{R}^n$ its only equivalent element within the unit cube. In what follows we will find it convenient to give $\mathbb{R}^n$ a Banach space structure by introducing two different norms, the ordinary Euclidean one $\|x\| = [\sum_{i=1}^n |x_i|^2]^{1/2}$ and the further norm $\|\cdot\|_E$ defined later.

That being stated, let $[M]$ be an arbitrary non-singular $n \times n$ matrix with integer entries and consider the linear invertible transformation M of $\mathbb{R}^n$ onto itself having $[M]$ as representative matrix with respect to the canonical base. The maps T of the n-torus we study here are defined by $T(x) = M(x) \bmod [0, 1[^n\ \forall x \in \mathbb{T}^n$ and always preserve the Haar–Lebesgue measure on $\mathbb{T}^n$. $[M]$ is known as the associated matrix of T.

Whenever $[M] \in SL(\mathbb{Z}, n)$ a simple algebraic manipulation shows that T is a one-to-one map of $\mathbb{T}^n$ onto itself and that the probability measure μ is invariant for both T and T^{-1}. The map T is then called an algebraic toral automorphism on $\mathbb{T}^n$, some properties of which, like ergodicity with respect to the invariant measure μ and hyperbolicity, are directly related to the spectrum of $[M]$ [14, 29]. Hyperbolicity of T on $\mathbb{T}^n$ is also equivalent to hyperbolicity of the linear mapping M on the covering space $\mathbb{R}^n$ [14]. The adjoint operator $\tilde{M}$ of M is hyperbolic on $\mathbb{R}^n$ as well. Let us focus our attention on $\tilde{M}$, which will be actually involved in the estimates later. Hyperbolicity of $\tilde{M}$ means that there exist two non-trivial linear subspaces of $\mathbb{R}^n$, $\boldsymbol{E}^s$ and $\boldsymbol{E}^u$, and a positive constant $\nu < 1$ satisfying the following properties:

(i) $\boldsymbol{E}^s \oplus \boldsymbol{E}^u = \mathbb{R}^n$;

(ii) $\tilde{M}(\boldsymbol{E}^s) = \tilde{M}^{-1}(\boldsymbol{E}^s) = \boldsymbol{E}^s$; $\tilde{M}(\boldsymbol{E}^u) = \tilde{M}^{-1}(\boldsymbol{E}^u) = \boldsymbol{E}^u$;

(iii) $k \in \boldsymbol{E}^s \Rightarrow \|\tilde{M}^m k \leqslant \nu^m \|k\| \, \forall m \in \mathbb{N}$; $k \in \boldsymbol{E}^u \Rightarrow \|\tilde{M}^{-m} k\| \leqslant \nu^m \|k\| \, \forall m \in \mathbb{N}$.

$\boldsymbol{E}^s$ is known as the stable space—or also the stable manifold of the only fixed point 0—whereas $\boldsymbol{E}^u$ is the unstable space—or unstable manifold, respectively. Conditions (ii) and (iii) lead to the further bounds

$$\begin{aligned} \|\tilde{M}^{-m} k\| &\geqslant \nu^{-m} \|k\| \qquad \forall k \in \boldsymbol{E}^s, m \in \mathbb{N} \\ \|\tilde{M}^{m} k\| &\geqslant \nu^{-m} \|k\| \qquad \forall k \in \boldsymbol{E}^u, m \in \mathbb{N}. \end{aligned} \tag{2.1}$$

Moreover, the decomposition (i) allows us to introduce another norm on $\mathbb{R}^n$, $\|\cdot\|_E$, which will be useful later. Since for every $k \in \mathbb{R}^n$ there are a unique vector $k_s \in \boldsymbol{E}^s$ and a unique $k_u \in \boldsymbol{E}^u$ such that $k = k_s + k_u$, the relationship below defines the desired norm

$$\|k\|_E = \|k_s\| + \|k_u\| \tag{2.2}$$

whose equivalence with respect to $\|\cdot\|$ ensures the existence of constants $\Lambda_E^-, \Lambda_E^+ > 0$ such that

$$\Lambda_E^- \|k\|_E \leqslant \|k\| \leqslant \Lambda_E^+ \|k\|_E \qquad \forall k \in \mathbb{R}^n \tag{2.3}$$

according to the definition of norm equivalence. Although any ergodic algebraic automorphism of $\mathbb{T}^n$ is also strong mixing, so that for $n > 4$ there are non-hyperbolic mixing automorphisms [30], our discussion about decay of correlations will be confined to hyperbolic automorphisms only.

In contrast, as a simple algebraic investigation shows, the case $|\det[M]| = d \neq 1$ corresponds to a d-to-one map T of $\mathbb{T}^n$ onto itself which still preserves the Lebesgue measure. It will be referred to as an algebraic toral endomorphism. Also in this non-invertible hypothesis T satisfies a mixing property if and only if its associated matrix $[M]$ has no root of unity as an eigenvalue [14]. Nevertheless, we will confine ourselves to the following cases only:

(a) algebraic toral endomorphisms whose tangent map M is hyperbolic on $\mathbb{R}^n$;

(b) purely expanding algebraic toral endomorphisms.

The endomorphism T has hyperbolic tangent map if and only if all of the eigenvalues of the associated matrix $[M]$ lie outside the unit circle, but there are eigenvalues λ_+ and λ_- satisfying $|\lambda_+| > 1$ and $|\lambda_-| < 1$. Whenever any eigenvalue of $[M]$ has modulus greater than 1 we say that the endomorphism is purely expanding. Of course, both classes of endomorphisms are mixing.

2.1. Analytic observables on $\mathbb{T}^n$

An observable on $\mathbb{T}^n$ is any function of the linear space $L^2(\mathbb{T}^n, \mathcal{B}, \mu)$ endowed with the scalar product (0.1) and with the induced L^2-norm $\|f\|_2 = (\langle f|f\rangle)^{1/2}$. We denote with $a \cdot b$ the usual inner product of vectors $a, b, \in \mathbb{R}^n$, i.e. the sum $\sum_{i=1}^n a_i b_i$ where $a_i \in \mathbb{R}$ stands for the ith component of a with respect to the canonical base. Characters $e_k(x) = \mathrm{e}^{\mathrm{i}2\pi k\cdot x}$, $k \in \mathbb{Z}^n$, $x \in \mathbb{T}^n$, provide a complete orthonormal set of $L^2(\mathbb{T}^n, \mathcal{B}, \mu)$, so that any observable f can be expanded into the Fourier series $f(x) = \sum_{k\in\mathbb{Z}^n} c_k(f) e_k(x)$, convergent with respect to the L^2-norm. Smooth or analytic observables can be regarded as periodic functions on the covering space $\mathbb{R}^n$, of period 1 on each variable x_i. The Dirichlet theorem ensures the Fourier series to be convergent not only with respect to the L^2-norm but also pointwise on $\mathbb{T}^n$. In both cases the regularity of f implies a fast decay of Fourier coefficients as $\|k\| \to \infty$. More precisely, it is well known that if f is analytic on $\mathbb{T}^n$ there are constants

$\alpha, \beta > 0$ such that $|c_k(f)| \leqslant \alpha\, \mathrm{e}^{-\beta\|k\|}\ \forall k \in \mathbb{Z}^n$, leading to an exponential decay of the Fourier spectrum.

2.2. *Decay of correlations for characters*

The explicit computation of correlations for arbitrary observables is a formidable task from an analytical and a numerical point of view, and can be performed successfully in some special cases only [28, 31, 32]. The simplest result concerns correlations between characters and is easily achieved by noting that the associated Koopman operator of an algebraic toral auto- or endomorphism maps the lattice $\mathbb{Z}^n$ onto itself. We have in fact, $\forall k \in \mathbb{Z}^n$, $x \in \mathbb{T}^n$ and $s \in \mathbb{N}$, the equality $(U^s e_k)(x) = e_k(T^s(x)) = \mathrm{e}^{\mathrm{i}2\pi k\cdot T^s(x)}$, which in the covering space also reads $(U^s e_k)(x) = \mathrm{e}^{\mathrm{i}2\pi k\cdot M^s x} = \mathrm{e}^{\mathrm{i}2\pi \tilde{M}^s k\cdot x}$, on having introduced the adjoint operator $\tilde{M}$ of M. As a consequence, for every $h, k \in \mathbb{Z}^n$ we get

$$\langle e_h|U^s e_k\rangle = \int_{[0,1[^n} \mathrm{e}^{-\mathrm{i}2\pi h\cdot x}\, \mathrm{e}^{\mathrm{i}2\pi \tilde{M}^s k\cdot x}\, \mathrm{d}\mu(x) = \delta_{h,\tilde{M}^s k} \tag{2.2.1}$$

with $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise, for any $a, b \in \mathbb{Z}^n$. Equation (2.2.1) allows to deduce estimates on the correlation decay of regular observables from the dynamical properties of the linear mapping $\tilde{M}$ on $\mathbb{Z}^n$.

2.3. *Decay of correlations for analytic observables*

In this section we prove that the correlation decay of any analytic observable on $\mathbb{T}^n$ is superexponential. More precisely we have the following theorem.

Theorem 2.1. Let T belong to one of the following classes of algebraic toral maps:

(a) hyperbolic automorphisms of $\mathbb{T}^n$;
(b) endomorphisms of $\mathbb{T}^n$ with hyperbolic tangent map;
(c) purely expanding endomorphisms of $\mathbb{T}^n$.

Then for any analytic observable f on $\mathbb{T}^n$ constants $A, B > 0$ and $R > 1$ exist such that

$$|C_s(f, f)| \leqslant A\, \mathrm{e}^{-BR^s} \qquad \forall s \in \mathbb{N} \tag{2.3.1}$$

with R dependent on T only. □

Proof. Let f be an arbitrary observable on $\mathbb{T}^n$ and $s \in \mathbb{N}$. By the continuity of the scalar product and of the (unitary) Koopman operator with respect to the L^2-norm we can write

$$\langle f|U^s f\rangle = \left\langle \sum_{h\in\mathbb{Z}^n} c_h(f) e_h \Big| U^s \left[\sum_{k\in\mathbb{Z}^n} c_k(f) e_k\right]\right\rangle = \sum_{h,k\in\mathbb{Z}^n} \overline{c_h(f)} c_k(f) \langle e_h|U^s e_k\rangle \tag{2.3.2}$$

and inserting (2.2.1) we obtain

$$\langle f|U^s f\rangle = \sum_{h,k\in\mathbb{Z}^n} \overline{c_h(f)} c_k(f) \delta_{h,\tilde{M}^s k} = \sum_{k\in\mathbb{Z}^n} \overline{c_{\tilde{M}^s k}(f)} c_k(f).$$

On the other hand, there also holds

$$\langle f|1\rangle\langle 1|f\rangle = \langle f|e_0\rangle\langle e_0|f\rangle = \overline{c_0(f)} c_0(f)$$

and the correlation reduces to

$$C_s(f, f) = \langle f|U^s f\rangle - \langle f|1\rangle\langle 1|f\rangle = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \overline{c_{\tilde{M}^s k}(f)} c_k(f).$$

The fundamental upper bound to autocorrelations will then be $|C_s(f,f)| \leqslant \sum_{k\in\mathbb{Z}^n\setminus\{0\}} |c_{\tilde{M}^s k}(f)||c_k(f)|$, provided that the series on the right-hand side converges. If f is analytic its spectrum decays exponentially and the previous bound becomes

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{k\in\mathbb{Z}n\setminus\{0\}} \mathrm{e}^{-\beta(\|\tilde{M}^s k\|+\|k\|)} \tag{2.3.3}$$

for some constants $\alpha, \beta > 0$, so that the behaviour on $s \in \mathbb{N}$ of objects like $\|k\| + \|\tilde{M}^s k\|$ is crucial in order to establish the desired estimates to $C_s(f,f)$. To this end we discuss separately cases (a), (b) and (c).

(a) *Hyperbolic automorphisms*. Suppose that the map T is a hyperbolic automorphism of $\mathbb{T}^n$. This implies, in particular, that the linear transformation M defines a one-to-one mapping of the lattice $\mathbb{Z}^n$ onto itself, and so does $\tilde{M}$. Let f be an analytic $L^2(\mathbb{T}^n, \mathcal{B}, \mu)$ function, for which therefore (2.3.3) holds, and suppose for simplicity that the index $s \in \mathbb{N}$ of the correlation $C_s(f,f)$ is even. By introducing the change of variable $h = \tilde{M}^{s/2}k$, the bound (2.3.3) is put into the following equivalent form:

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{h\in\mathbb{Z}^n\setminus\{0\}} \mathrm{e}^{-\beta(\|\tilde{M}^{s/2}h\|+\|\tilde{M}^{-s/2}h\|)}. \tag{2.3.4}$$

Recalling the definitions and notation concerning the hyperbolic structure of $\tilde{M}$, we obtain

$$\frac{1}{\Lambda_E^-}(\|\tilde{M}^{s/2}h\| + \|\tilde{M}^{-s/2}h\|) \geqslant \|\tilde{M}^{s/2}h_s + \tilde{M}^{s/2}h_u\|_E + \|\tilde{M}^{s/2}h_s + \tilde{M}^{-s/2}h_u\|_E \tag{2.3.5}$$

but since $\tilde{M}^{s/2}h_s, \tilde{M}^{-s/2}h_s \in \boldsymbol{E}^s$ and $\tilde{M}^{s/2}h_u, \tilde{M}^{-s/2}h_u \in \boldsymbol{E}^u$ we can rewrite the right-hand side as

$$\|\tilde{M}^{s/2}h_s\| + \|\tilde{M}^{s/2}h_u\| + \|\tilde{M}^{-s/2}h_s\| + \|\tilde{M}^{-s/2}h_u\| \geqslant \nu^{-s/2}\frac{1}{\Lambda_E^+}\|h\|. \tag{2.3.6}$$

As a conclusion $\|\tilde{M}^{s/2}h\| + \|\tilde{M}^{-s/2}h\| \geqslant \nu^{-s/2}\|h\|\Lambda_E^-/\Lambda_E^+$. The case of odd $s \in \mathbb{N}$ is treated in a completely similar way, by posing $h = \tilde{M}^{(s-1)/2}k$ within (2.3.3), and the result reads $\|\tilde{M}^{(s+1)/2}h\| + \|\tilde{M}^{-(s-1)/2}h\| \geqslant \nu^{-(s-1)/2}\|h\|\Lambda_E^-/\Lambda_E^+$. We now simply replace into (2.3.4) and conclude

$$|C_s(f,f)| \leqslant \alpha^2\, \mathrm{e}^{-\nu^{-\lfloor s/2\rfloor}\beta\Lambda_E^-/\Lambda_E^+} \sum_{h\in\mathbb{Z}^n\setminus\{0\}} \mathrm{e}^{-\nu^{-\lfloor s/2\rfloor}\beta(\|h\|-1)\Lambda_E^-/\Lambda_E^+} \qquad \forall s \in \mathbb{N} \tag{2.3.7}$$

where $\lfloor x \rfloor$ stands for the integer part of $x \in \mathbb{R}$ and the residual series is bounded by a constant independent on s.

(b) *Endomorphisms with hyperbolic tangent map*. This case can be treated like the previous one. We only have to notice that now $\tilde{M}$ defines a transformation of $\mathbb{R}^n$ which is still one-to-one but not onto. As a consequence $\tilde{M}(\mathbb{Z}^n\setminus\{0\}) \subset \mathbb{Z}^n\setminus\{0\}$. Consider, for instance, a correlation $C_s(f,f)$ with f analytic and even $s \in \mathbb{N}$. By the change of variables $h = \tilde{M}^{s/2}k$, which is well defined, and due to the hyperbolic structure of $\tilde{M}$, (2.3.3) can be written as

$$|C_s(f,f)| \leqslant \alpha^2 \sum_{h\in\tilde{M}^{s/2}(\mathbb{Z}^n\setminus\{0\})} \mathrm{e}^{-\beta(\|\tilde{M}^{s/2}h\|+\|\tilde{M}^{-s/2}h\|)}. \tag{2.3.8}$$

An analogous estimate holds for odd s by introducing the change of variable $h = \tilde{M}^{(s-1)/2}k$. The same bounds on $\|\tilde{M}^{-s/2}h\| + \|\tilde{M}^{s/2}h\|$ and $\|\tilde{M}^{(s+1)/2}h\| + \|\tilde{M}^{-(s-1)/2}h\|$ previously established lead then to the inequality

$$|C_s(f,f)| \leqslant \alpha^2\, \mathrm{e}^{-\nu^{-\lfloor s/2\rfloor}\beta\Lambda_E^-/\Lambda_E^+} \sum_{h\in\mathbb{Z}^n\setminus\{0\}} \mathrm{e}^{-\nu^{-\lfloor s/2\rfloor}\beta(\|h\|-1)\Lambda_E^-/\Lambda_E^+} \qquad \forall s \in \mathbb{N}. \tag{2.3.9}$$

(c) *Expanding endomorphisms*. Expanding endomorphism means that all of the eigenvalues of the adjoint operator $\tilde{M}$ have modulus greater than 1. Therefore, a positive constant $\nu < 1$ will exist such that $\|\tilde{M}^{-1}k\| \leqslant \nu\|k\| \,\forall k \in \mathbb{R}^n$ or, equivalently, $\|\tilde{M}k\| \geqslant \nu^{-1}\|k \,\forall\|k \in \mathbb{R}^n$ which implies, in particular, $\|\tilde{M}^s k\| + \|k\| \geqslant \nu^{-s}\|k\| + \|k\| = (\nu^{-s}+1)\|k\|$.

The upper bound (2.3.3) for the autocorrelation $C_s(f, f)$ of an analytic observable f will be written, for each $s \in \mathbb{N}$ and with no change of variables, as

$$|C_s(f, f)| \leqslant \alpha^2 \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \mathrm{e}^{-\beta(\nu^{-s}+1)\|k\|} = \alpha^2\, \mathrm{e}^{-\beta(\nu^{-s}+1)} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \mathrm{e}^{-\beta(\nu^{-s}+1)(\|k\|-1)} \tag{2.3.10}$$

with the usual bounded residual series. The proof is complete. □

Remark. In the particular situation that the complexification of the linear operator $\tilde{M}$ can be diagonalized on $\mathbb{C}^n$, a small modification of the previous discussion provides a more specific characterization of the decay rates. For simplicity's sake, let us denote with the same symbol $\tilde{M}$ the complexification of $\tilde{M}$. Let T be a hyperbolic automorphism and suppose then that $\tilde{M}$ admits the base $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ of eigenvectors on $\mathbb{C}^n$ with corresponding—possibly complex or coinciding—eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, none of which lies on the unit circle. Any vector $h \in \mathbb{R}^n$ will be written in a unique way as

$$h = \sum_{i=1}^{n} c_i(h)u_i \qquad c_i(h) \in \mathbb{C} \;\forall i = 1, 2, \ldots, n \tag{2.3.11}$$

and a norm $\|\cdot\|_u$ will be defined by $\|h\|_u = \sum_{i=1}^n |c_i(h)|$, equivalent to the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$ according to $\Lambda_u^-\|h\|_u \leqslant \|h\| \leqslant \Lambda_u^+\|h\|_u \,\forall h \in \mathbb{R}^n$, $\Lambda_u^-, \Lambda_u^+ > 0$. We have

$$\|\tilde{M}^{s/2}h\| + \|\tilde{M}^{-s/2}h\| \geqslant \operatorname*{Inf}_j\, [|\lambda_j|^{s/2} + |\lambda_j|^{-s/2}] \frac{\Lambda_u^-}{\Lambda_u^+}\|h\| \tag{2.3.12}$$

where $\mathrm{Inf}_j[|\lambda_j|^{s/2}+|\lambda_j|^{-s/2}]$ increases exponentially as $s \to +\infty$. An analogous calculation holds for odd s and provides

$$\|\tilde{M}^{(s+1)/2}h\| + \|\tilde{M}^{-(s-1)/2}h\| \geqslant \operatorname*{Inf}_j\, [|\lambda_j|^{1/2}(|\lambda_j|^{s/2} + |\lambda_j|^{-s/2})] \frac{\Lambda_u^-}{\Lambda_u^+}\|h\| \tag{2.3.13}$$

with the same conclusion. □

Since estimates (2.3.12) and (2.3.12) obviously extend to the case of an endomorphism T with hyperbolic tangent map, the same characterization of decay rates also holds.

As for expanding endomorphisms, we can establish a very simple relation between the computed expansion rate ν^{-1} and the eigenvalues of the linear operator. Indeed by using (2.3.11) we obtain

$$\|\tilde{M}^s h\| + \|h\| \geqslant \Lambda_u^-(\|\tilde{M}^s h\| + \|h\|_u) \geqslant \frac{\Lambda_u^-}{\Lambda_u^+}\left(\left[\operatorname*{Inf}_j |\lambda_j|\right]^s + 1\right)\|h\| \tag{2.3.14}$$

on having $\mathrm{Inf}_j|\lambda_j| > 1$.

It is important to compare our estimates with those described in [16] for the cat map. There the decay of correlations for analytic observables is characterized in a weaker way than in the present work; correlation decay turns out to be more than exponential, whereas we can state a more precise superexponential decay law. The systematic construction of observables obeying a power-decay law, even if not explicitly explained here, can be performed as well and so can be the estimate of correlation decay for smooth observables. Nevertheless, if λ denotes the eigenvalue of $\tilde{M}$ with modulus greater than one, the exponential decay rate computable for a C^q observable by the spectral methods presented here turns out to be $q\log|\lambda|/2$, exactly one half the value $q\log|\lambda|$ found in [16]. That better result is not astonishing, since it lies on a very detailed characterization of the orbits of $\tilde{M}$ on the reciprocal lattice $\mathbb{Z}^2$, followed by an ingenious, suitable rearrangement of such orbits. The extension of the same arguments to other algebraic automorphisms of the 2-torus and to higher-dimensional auto- and endo-morphisms is certainly non-trivial and fairly cumbersome, in spite of the simplicity and generality of the present analysis. In both cases, the spectral method reveals its capability to provide a clear relation between smoothness of the observable and estimated decay rate.

3. Skew-endomorphisms of the 2-torus with Bernoulli base

Spectral methods discussed in section 1, and in particular the kind of estimates used in the proof of proposition 1.4, can be fruitfully applied to a class of skew-endomorphisms T of the 2-torus defined by the relationship

$$T:\begin{cases} x' = px \bmod [0,1[\\ y' = y+\omega+\varepsilon x \bmod [0,1[\qquad \forall (x,y)\in[0,1[^2 \end{cases} \tag{3.1}$$

where $p\in\mathbb{Z}\backslash\{-1,0,1\}$ and $\varepsilon,\omega\in\mathbb{R}$. The 2-torus is parametrized by $[0,1[^2$ and endowed with the invariant Lebesgue–Haar measure μ on the σ-field of Borel sets. For simplicity's sake, we will confine ourselves to the mixing—and actually exact [18]—case, which is known to occur if and only if $\varepsilon\in\mathbb{R}\backslash\mathbb{Q}$ [17, 19]. The (complete) orthonormal set in L^2 we consider is the usual Fourier base $e_{a,b}=\mathrm{e}^{\mathrm{i}2\pi(ax+by)}\ \forall(a,b)\in\mathbb{Z}^2$ as in the previous case of toral algebraic endomorphisms. We have the following results.

Theorem 3.1. Let $\varepsilon\in\mathbb{R}$ satisfy a Diophantine condition and let $f:\mathbb{T}^2\longrightarrow\mathbb{C}$ be analytic on $\mathbb{T}^2$. Then $\forall\gamma>0$ there exists a sequence $(a_{\gamma,s})_{s\in\mathbb{N}}\subset\mathbb{R}^+$ such that $\lim_{s\to+\infty}a_{\gamma,s}=0$ and

$$|C_s(f,f)|\leqslant a_{\gamma,s}s^{-\gamma} \qquad \forall s\in\mathbb{Z}^+. \tag{3.2}$$

□

Theorem 3.2. Let $f:\mathbb{T}^2\longrightarrow\mathbb{C}$ a C^r function on the 2-torus with $r\geqslant 2$ and ε a Diophantine number

$$\left|\varepsilon-\frac{m}{q}\right|^{-1}\leqslant \mu_\varepsilon|q|^{\gamma_\varepsilon} \qquad \forall q\in\mathbb{Z}\backslash\{0\}, m\in\mathbb{Z}. \tag{3.3}$$

□

Then $\forall\gamma>0$ such that $r\geqslant\lfloor 1+\gamma(\gamma_\varepsilon-1)\rfloor+1$ there exists a sequence $(a_{\gamma,s})_{s\in\mathbb{N}}\subset\mathbb{R}^+$ for which $\lim_{s\to+\infty}a_{\gamma,s}=0$ and (3.2) holds.

Sketch of the proof. The core of the proof is a full characterization of correlation decay between vectors of the orthonormal base [19]. By choosing two arbitrary characters, $e_{a,b}$ and $e_{c,d}$, $(a,b),(c,d)\in\mathbb{Z}^2$, the correlation $C_s(e_{a,b},e_{c,d})$ at time $s\in\mathbb{N}$ takes the form

$$C_s(e_{a,b},e_{c,d})=\delta_{b,d}\,\mathrm{e}^{\mathrm{i}\pi(2\omega+\varepsilon)ds}(-1)^{c+ap}\frac{\sin[\pi c+\frac{1}{2}\phi_s]}{\pi c+\frac{1}{2}\phi_s}\prod_{j=0}^{s}\frac{\sin(\frac{1}{2}p\phi_j)}{p\sin(\frac{1}{2}\phi_j)}-\delta_{a,0}\delta_{b,0}\delta_{c,0}\delta_{d,0} \tag{3.4}$$

where $\phi_j=2\pi[\varepsilon d(p-1)^{-1}-(a+\varepsilon d(p-1)^{-1})p^{-j}]$ and the real functions $x\to\sin x/x$, $x\to\sin(px)/(px)$ are also defined at $x=0$ by continuity.

It is then straightforward to verify that whenever ε is irrational and $b \neq 0$ the above correlations decay at an exponential rate, on having asymptotically in s

$$|\langle e_{a,b}|U^s e_{c,b}\rangle| \sim \left[\sin\left(\pi p \frac{\varepsilon b}{p-1}\right)\right]^s \left[p \sin\left(\pi \frac{\varepsilon b}{p-1}\right)\right]^{-s}. \qquad (3.5)$$

The decay rate approaches the critical value 1 when the distance of $\varepsilon b/(p-1)$ from $\mathbb{Z}$ tends to zero. All of the above features combine to provide the proof, which works like that of proposition 1.4. For a (even too) detailed proof and further analytical and numerical results about this three-parameter family of skew-endomorphisms we refer the reader to [19, 20].

□

4. Conclusions

For hyperbolic systems, spectral methods allow one to establish a relationship between the smoothness of the observable and its own (sometimes even superexponential) decay rate. In contrast, general techniques of symbolic dynamics foresee only an exponential decay rate which is independent of the smoothness of the observables and usually difficult to relate to the parameters of the map. Symbolic dynamics techniques are based on approximations of the observables by means of piecewise constant functions on the cylindrical sets of the associated Markov partition. The error introduced by this first approximation is exponentially small with respect to $s \in \mathbb{N}$ by assuming the additional requirement that observables be Hölder continuous. C^k, $k \in \mathbb{Z}^+$, or C^ω observables are certainly Hölder continuous, as Lipschitz continuous, but the estimate of the approximation error carries no trace of such a regularity. As a conclusion, spectral techniques, even if by paying the price of a lesser generality, offer the twofold advantage of a rather strict relation between the estimated decay rate and the smoothness of the observables, on the one hand, and between the decay rate and the system parameters on the other.

As for non-hyperbolic systems, a domain where general methods of symbolic dynamics are not available, spectral techniques can also provide quite satisfactory estimates to correlation decay of analytic or sufficiently smooth observables.

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